Road Vehicles—Friction Materials—Elastic Properties Measurements

RATIONALE

Not applicable.

FOREWORD

The elastic properties of friction materials are important design parameters because modulus variations may be significant in noise generation of braking systems. Variations in elastic properties as a result of processing and formulation are often a primary determinant of component performance. The purpose of this test procedure is to evaluate the elastic properties of friction materials in brake pads. Data are provided in a format that is directly useful for modeling analysis and simulation.

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1. SCOPE

This SAE Standard specifies a method for testing and measuring elastic constants in friction materials by precise ultrasonic velocity measurements.

2. REFERENCES (SEE ALSO REFERENCES OF APPENDIX A)

2.1 Application Publications

The following publications form a part of this specification to the extent specified herein.

Results of Friction Materials Elastic Constant Round Robin Study, Donald E. Yuhas and Paul Sanders, 25th annual Brake Colloquium, Orlando FL October, 2007 SAE # 07BC-40

Non-linear Aspects of Friction Material Elastic Constants, Donald E. Yuhas, Jim Ding, and Srikanth VenKatesan, 24th annual Brake Colloquium, Dallas TX October, 2006 SAE # 2006-01-3193

Elevated Temperature Measurements of Elastic Constants in Polymer Composites, Donald E. Yuhas and Bruce Isaacson, The 43rd International SAMPE Symposium May 31-June 4, 1998, Anaheim California

H.M. Ledbetter and D.T. Read, "Orthorhombic Elastic Constants of an NbTi/Cu composite Superconductor", Jour. App. Phys, #5 May, 1977

Every A.G., “Determination of the Elastic Constants of Anisotropic Solids”, NDT International Vol. 27, No. 1, p. 3, 1993

Disc Brake Squeal, Frank Chen, Chin An Tan, and Ronald L. Quaglia eds. Chapter 12 D.E. Yuhas and Marjorie P. Yuhas. Friction Material Elastic Constant Measurements ISBN 10 0-7680-1248-1

3. DEFINITIONS

In this test, the elastic constants of friction materials are determined by measuring both the longitudinal and shear wave speeds propagating at various directions in the friction material. It is important to record and maintain the relationship between the direction of ultrasonic propagation, wave polarization, and the principle axes of the friction material. From the measured ultrasonic velocities, and the symmetry of the friction material, the elastic constants as well as the Engineering constants i.e. Young’s moduli, Shear Moduli, and Poisson’s Ratios can be calculated.

3.1 Coordinate System Definition

The coordinate geometry referenced to the disk pad shape is shown in Figure 1. The same geometry is used for truck blocks and with a slight modification can be used for drum brakes. In all cases the “3” or ‘z” axis is the thickness direction and the “1” or “x” direction is along the rotation axis. This coordinate definition is used throughout the gathering of data and the subsequent analysis. Failure to adhere to the appropriate data collection protocol can lead to some unusual and potentially misleading results.

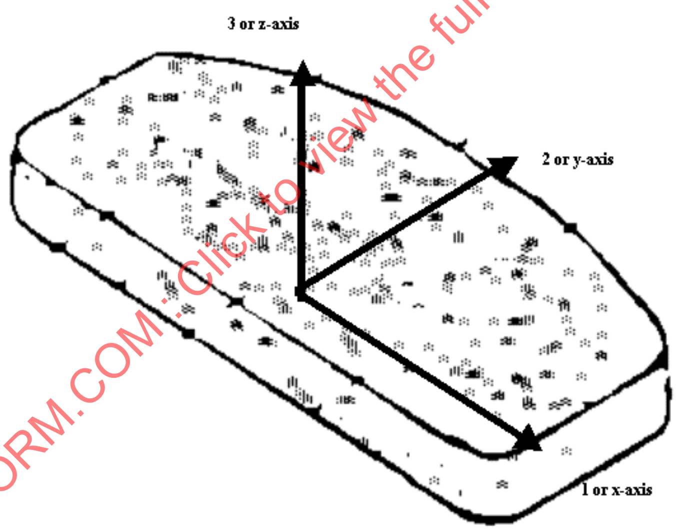

FIGURE 1 - BRAKE PAD BASED COORDINATE SYSTEM

In the coordinate system shown in Figure 1, the “3” axis is perpendicular to the friction surface, the “1” axis is along the long axis of the pad, and the “2” axis is parallel to the short axis of the pad.

3.2 Ultrasonic Modes

Ultrasonic – mechanical vibration in the frequency range from 0.1 to 100 MHz.

Longitudinal or compressional wave – ultrasonic wave mode where the direction of wave propagation is parallel to the particle displacement or polarization of the wave.

Shear wave – ultrasonic wave mode where the direction of wave propagation is perpendicular to the particle displacement or polarization.

Using the above definitions and the coordinate system defined in Figure 1, we can define the following wave modes:

V_{33} – Longitudinal mode propagation along the “3” direction with polarization along “3”
V_{11} – Longitudinal mode propagation along the “1” direction with polarization along “1”
V_{22} – Longitudinal mode propagation along the “2” direction with polarization along “2”
V_{31} – Shear mode propagating along the “3” direction polarized along the “1” direction
V_{32} – Shear mode propagating along the “3” direction polarized along the “2” direction
V_{21} – Shear mode propagating along the “2” direction polarized along the “1” direction
V_{12} – Shear mode propagating along the “1” direction polarized along the “2” direction

V_{45} is a shear mode propagating along j – k (the vector .707j - .707k) and polarized along j + k (the vector .707j +.707k). Here, j is the unit vector along the “2” (or 1) direction and k is the unit vector along the “3” direction. Using the same convention as used for the other modes along the principle axes, this mode would be designated as $V_{.707y-.707z,.707y+.707z}$.

3.3 Elastic Constants and Engineering Constants

The equations relating the measured ultrasonic velocities to the material elastic constants and the engineering constants are derived in Appendix A.

4. SYMBOLS AND UNITS

TABLE 1 - SYMBOLS AND UNITS

Parameter	Symbol	Equation	Unit
Initial friction material thickness	X_1, X_2, X_3		mm
Longitudinal Transit time	T_L		microseconds
Longitudinal offset	T_{Lo}		microseconds
Shear Transit time	T_S		microseconds
Shear offset	T_{So}		microseconds
Coupling Force	F_c		kilograms
Longitudinal velocity	V_{ii}	X_i/t	mm/microsecond
Shear Velocity	V_{ij} i =/ j	X_i/t	mm/microsecond
Density	ρ		g/cc
Unique axis Longitudinal Modulus	C_{33}	ρV^2_{33}	Gpa
In-plane Longitudinal Modulus	$C_{11} = C_{22}$	$\rho(<V_{11}, V_{22}>^2$	Gpa
In-plane Shear Modulus	C_{66}	$\rho(<V_{21}, V_{12}>^2$	Gpa
Out-of-plane Shear Modulus	$C_{44} = C_{55}$	$\rho(<V_{31}, V_{32}, V_{13}, V_{23}>^2$	Gpa
First off-diagonal Elastic Constant	C_{12}	$c_{12} = \hat{c}_{11} - 2\hat{c}_{66}$	Gpa
Second off-diagonal Elastic Constant	C_{13}	$c_{13} = \sqrt{\left[\left(2\rho V_{45s}^{2} - 0.5\left(c_{22} + c_{33} + 2c_{44}\right)\right)^2 - .25\left(c_{33} - c_{22}\right)^2\right]} - c_{44}$	Gpa

5. EQUIPMENT

The test equipment should include:

a. Ultrasonic pulser-receiver unit

b. Waveform digitizer and display

c. Coupling load test fixture

d. 2 longitudinal wave ultrasonic sensors

e. 2 shear wave ultrasonic sensors

f. Micrometer

g. Balance

h. Ultrasonic couplant

i. Propagation Timing Standard

6. SAMPLE PREPARATION

For friction materials, the material symmetry is transversely isotropic with the unique axis along the materials thickness direction (axis “3” in Figure 1). We require a minimum of one rectangular sample oriented along the principle axes of the pad and one sample which has been sliced at a 45-degree angle relative to the “unique” axis of the pad. Figure 2 shows the brake pad cutting diagram.

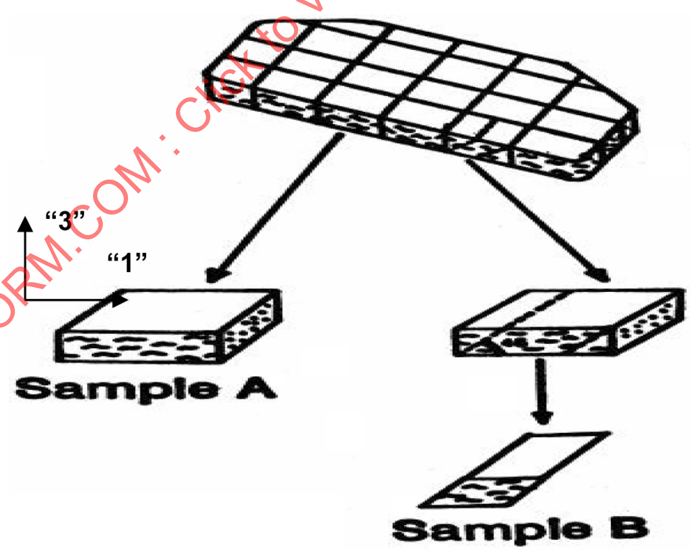

FIGURE 2 - TYPICAL CUTTING DIAGRAM FOR BRAKE PAD

6.1 Rectangular Samples

The first step in sample preparation is to remove the friction material from the steel backing using a band saw. The friction material is then sliced into a 20 mm wide segment parallel to the “2” direction. From this segment, small, 15 mm by 20 mm by ~8 mm rectangular test specimens are cut. Note that material is discarded from areas near the injection molding sites. Saw cut marks are removed using a belt-sander and efforts are made to produce rectangular test pieces with parallel surfaces. The longest sample dimension, (20 mm), corresponds to the longest dimension of the original pad (“1=x” direction in our defined coordinate system). The smallest dimension (~8 mm) always corresponds to the thickness dimension (“3=z” direction in our defined coordinate system). The parallelism is important for all samples and we generally want to control the variation to under ±25 microns. The above shows recommended sizes for typical automotive disc brake pad. For larger truck blocks it may be desirable to increase the sample size in proportion by a factor of as much as 3 times. The thickness direction (“3” axis) is the controlling dimension. For drum brake segments, the same rectangular pieces are cut from the segment, however in this case the slight curvature of the surface must be removed by sanding. This is particularly important for the concave surface.

6.2 45-degree Samples

A second sample type, a cut 45-degree relative to the thickness direction, is used to obtain the off-diagonal elements to the elastic constant matrix. This is shown as Sample B in Figure 2. This Figure is somewhat misleading in that it suggests that Sample B, (45-degree Sample) has a large surface area for the face normal to the “3” axis. The 45-degree sample must be “thinner” than the thickness of the pad from which it came, i.e. thinner than the “3” direction thickness. Typical thickness is 6 mm. Measurements are made on this piece by propagation of waves perpendicularly to the 45-degree cut surface as indicated by the white arrow in Figure 3. This sample must be cut such that the 45-degree faces are parallel to one another and the ultrasound can propagate perpendicular to this cut surface reaching the other side before being reflected and refracted by a sidewall.

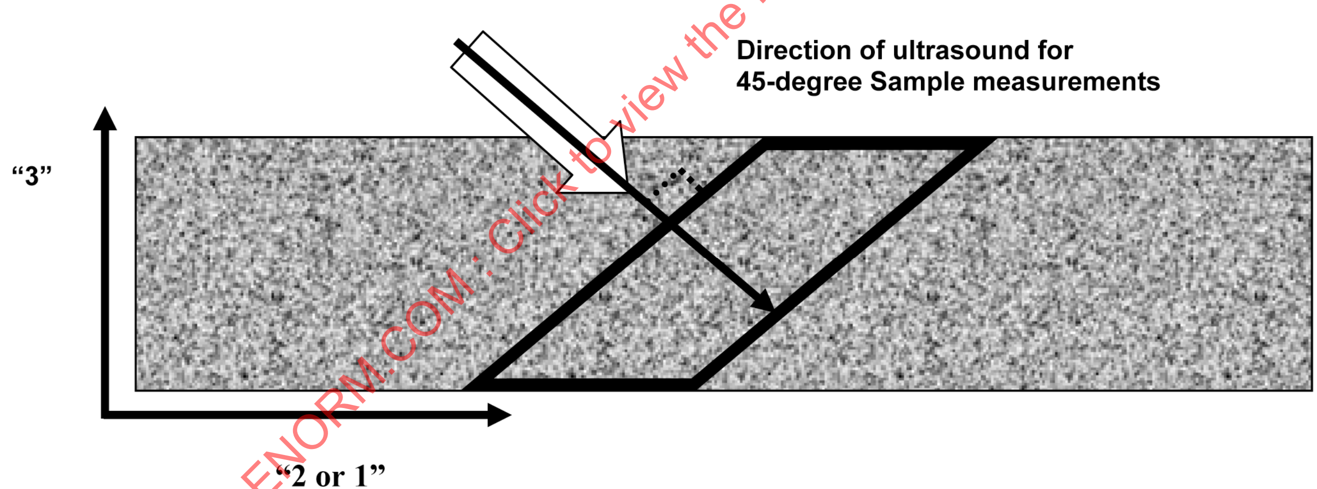

FIGURE 3 - DIRECTION OF ULTRASOUND FOR 45-DEGREE SAMPLE MEASUREMENTS - PROPAGATION IS NORMAL TO THE 45-DEGREE CUT SURFACE

Figure 3 shows a friction material in the 3-2 (or 3-1) plane with a dark outline showing a 45-degree cut, which is appropriate for the transit time measurement.

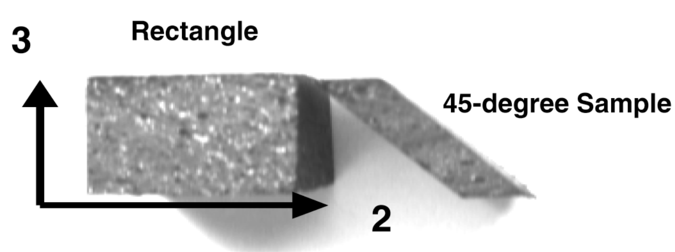

FIGURE 4 - CUT RECTANGULAR AND 45-DEGREE SAMPLE

Figure 4 shows a picture of a typical rectangular and 45-degree specimen. The thickness of this piece along the “3” axis is generally in the range from 6 mm to 10 mm. The 45-degree sample has been oriented so that it appears as it was in the pad prior to slicing. Its thickness, measured normal to the 45-degree cut surfaces, is only ~6 mm to 7.5 mm. All coordinate axis designations are referenced to the coordinate system shown in Figure 1.

For all samples the parallelism should be accurate to ±25 microns. The surface finish as obtained with 200 grit sandpaper. For the 45-degree sample, the angular uncertainty is controlled primarily by the cutting fixture. It should be better than ±5 degrees.

7. EQUIPMENT AND TEST RIG SPECIFICATION

Ultrasonic pulser-receiver unit must have a frequency bandwidth of 90 MHz, be capable of shock excitation of transducer in the frequency range from range of 1 to 10 MHz, and have a dynamic range of more than 90 dB.

Waveform digitizer and display must have an acquisition speed greater than 50 MHz., ability to display of waveforms in real-time, have automatic peak detection, signal averaging, (50 times minimum) and automatic transit time measurement capability. Timing precision of ±10 nanoseconds is required.

The coupling load test fixture must be capable of supplying and monitoring up to 1000 N (220 lb) of compressive load to the samples and of maintaining the polarization of the ultrasonic sensors to better than = ±5 degrees. The precision of the load fixture should be ±20 N (5 lb). For shear wave transducers, the direction of polarization is indicated on the sensors. The fixture must allow for rotation of each sensor (about the propagation direction) to insure that the polarization of the sensors are co-linear.

The longitudinal wave ultrasonic sensors must have a center frequency in the range from 1 to 5 MHz and a 10 dB bandwidth greater than 80% and have an active piezoelectric element less than 1.25 cm in diameter.

The shear wave transducers must be linearly polarized with polarization clearly marked. They must operate with a center frequency in the range from 1 to 5 MHz and have a 10 dB bandwidth greater than 80%, and have an active piezoelectric element less than 1.25 cm in diameter.

A propagation timing standard with known (independently measured) transit times for both longitudinal and shear waves must be accurate to ±5 nanoseconds.

The balance must have precision better than .01 g.

8. CALIBRATION

The calibration of the system is critical to achieving accurate results. Because of unknown triggering delays in the instrumentation and propagation delays in the piezoelectric sensors, it is necessary to calibrate the system using a test sample where the propagation time is known or measured independently. The ultrasonic system must be supplied with a steel calibration block whose transit time for both longitudinal and shear waves has been measured independently (See Figure 5).

Performing a calibration involves making a series of transit time measurements using both shear and compressional wave transducers. These measurements are then compared to those of a steel calibration sample and an offset correction factor, T^s_{Offset} (calibration factor) is subtracted from each subsequent transit time measurement. There will in general be different calibration factors for the shear and the compressional waves. Calibration should be performed at the beginning of each measurement session.

$$T^s_{offset} = T^s_{measured} - T^s_{standard} \quad \text{(Eq. 1)}$$

where

$T^s_{measured}$ is the measured shear wave transit time through the standard and
$T^s_{standard}$ is the independently measured shear transit time of the standard

$$T^{C}_{offset} = T^{C}_{measured} - T^{C}_{standard} \qquad \text{(Eq. 2)}$$

where

$T^{s}_{measured}$ is the measured longitudinal wave transit time through the standard and
$T^{s}_{standard}$ is the independently measured longitudinal transit time of the standard

Figure 5 shows the steel calibration block sandwiched between 2 transducers. Because the steel is a linear ultrasonic material the transit time is independent of coupling load. However, for consistency the transit time is measured using a coupling pressure of 4 MPa, the same load as that used to measure the friction material.

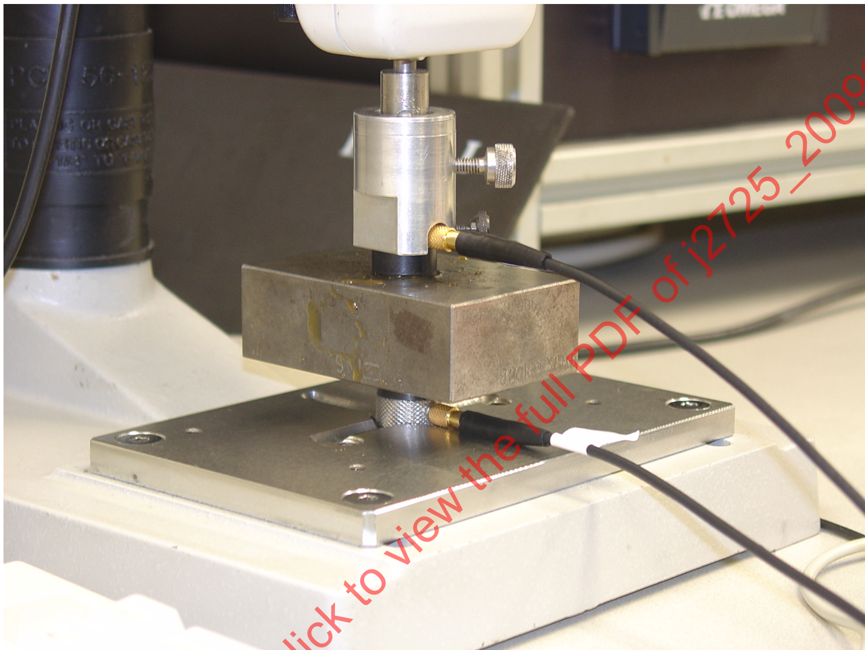

FIGURE 5 - CONFIGURATION FOR MEASURING THE TRANSIT TIME IN A STEEL CALIBRATION TEST SAMPLE

9. TEST PROCEDURE (ELASTIC CONSTANTS)

9.1 Ensure shear wave and longitudinal wave calibration factors, (T^{C}_{offset}, T^{S}_{offset}) have been measured.

9.2 Prepare rectangular and 45-degree cut samples as described previously. Because of the non-uniformity of the friction material, it is recommended that a minimum of 3 rectangles and 3, 45-degree samples be tested.

9.3 Measure sample dimensions X_1, X_2, and X_3.

9.4 Weigh the rectangular piece and calculate the density.

9.5 Using the longitudinal transducers measure transit times in the V_{33}, V_{22}, and V_{11} orientations. Apply calibration factors and compute the velocity.

For linear elastic materials the transit time is independent of coupling load. This is the case for longitudinal modes with propagation components in the plane of the pad (V_{11} and V_{22}). To measure these modes, one only needs to supply sufficient load to remove excess couplant from the layer between sample and ultrasonic sensor. However, for the V_{33} mode, there is a noticeable load dependence. This load dependence complicates the measurement process. At low coupling loads, the measured velocity for the V_{33} mode can exhibit significant variability and it may be difficult to identify the correct measurement point. Both measurement variation as well as the ease of identifying the appropriate measurement point is improved at increased load. It is recommended that the V_{33} mode be measured at a coupling pressure of 4 MPa (580 psi). This value is somewhat arbitrary but appears to be sufficiently high so that the measurement point can be identified unambiguously. Figure 6 shows a typical sample mounting and loading configuration.

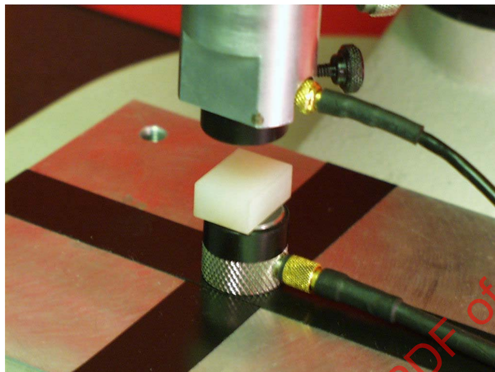

FIGURE 6 - RECTANGULAR SAMPLE MOUNTED IN COMPRESSION FIXTURE

Figure 6 shows rectangular sample sandwiched between two ultrasonic transducers in a fixture capable of delivering a coupling load of ~ 4 MPa.

9.6 Using the shear transducers measure transit times in the V_{31}, V_{32}, V_{21}, and V_{45} orientations. Apply calibration factors and compute the velocity.

For linear elastic materials the transit time is independent of coupling load. This is the case for the shear mode with propagation components in the plane of the pad (V_{21}). To measure these modes, one only needs to supply sufficient load to remove excess couplant from the layer between sample and ultrasonic sensor. However, for the V_{31}, V_{32}, and V_{45} modes, there is a noticeable load dependence. Unlike the longitudinal mode V_{33} which can exhibit significant pulse-shape variation with load, the shear wave pulses are undistorted. Thus there is no ambiguity in determining the measurement point. However, the velocity does depend on load so the load needs to be controlled and monitored. For consistency, for these shear modes we have standardized on the same coupling pressure as that used for V_{33} (4 MPa).

9.7 Using Equations 14, 15, 16, and 17 (from Appendix A) calculate the elastic constants. Use Equation 18 to calculate the engineering constants.

10. RECOMMENDED INFORMATION TO INCLUDE IN ELASTIC CONSTANT REPORT

10.1 Sample Information

a. Manufacturer, identification of brake lining, material batch reference number

b. Sample dimensions with measurement error

c. Number of samples tested

10.2 Test Conditions and Hardware

a. Transducer frequency and serial number

b. Transit time standard, shear and longitudinal offsets

10.3 Raw Data

Tabular data identifying the ultrasonic velocity mode, thickness, measured transit time, corrected (calibrated) transit time, computed velocity and load at measurement. It is recommended that three repeat measurements be made for each mode.

RAW TRANSIT TIME DATA

Mode	Corrected Time	Coupling Pressure	Thickness	Velocity	
	µsec	(MPa)	mm	km/sec	
v33	6.527	4	9.3472	1.432	
v33	6.527	4	9.3472	1.432	
v33	6.487	4	9.3472	1.441	
v33	6.487	4	9.3472	1.441	% st Dev
			Avg	1.435	0.362
v22	5.627	1	14.986	2.663	
v22	5.647	1	14.986	2.654	
v22	5.627	1	14.986	2.663	
			Avg	2.660	0.195
v11	7.487	1	20.2946	2.711	
v11	7.487	1	20.2946	2.711	
v11	7.487	1	20.2946	2.711	
			Avg	2.711	0.000
v31	8.383	4	9.3472	1.115	
v31	8.423	4	9.3472	1.110	
v31	8.383	4	9.3472	1.115	
v31	8.383	4	9.3472	1.115	
			Avg	1.113	0.259
v32	8.503	4	9.3472	1.099	
v32	8.463	4	9.3472	1.104	
v32	8.443	4	9.3472	1.107	
v32	8.443	4	9.3472	1.107	
			Avg	1.103	0.366
v21	8.863	1	14.986	1.691	
v21	8.783	1	14.986	1.706	
v21	8.843	1	14.986	1.695	
			Avg	1.697	0.458
v45	4.723	4	5.4102	1.146	
v45	4.563	4	5.4102	1.186	
v45	4.703	4	5.4102	1.150	
v45	4.603	4	5.4102	1.175	
			Avg	1.164	1.585

FIGURE 7 - RAW ULTRASONIC AND TRANSIT TIME DATA

For each mode the average velocity and the percent standard deviation are calculated.

10.4 Averaged Data and Calculated Elastic and Engineering Constants

For transversely isotropic materials with the unique axis along the “3” direction, there are several redundant modes that are measured. This is done as a “sanity” check to verify the symmetry of the friction materials as well as to provide some averaging to reduce variability. It is recommended that the values of V_{11} and V_{22} be averaged and that the values of V_{31} and V_{32} be averaged. The resultant is five independent velocity measurements that are used to calculate both the elastic and engineering constants as outlined in Appendix A. A typical suggested format for the data is outlined below.

TABLE 2 - SUMMARY DATA FOR ULTRASONIC MEASUREMENTS

	V_{33}	$\langle V_{22};V_{11}\rangle$	$\langle V_{31};V_{32}\rangle$	V_{12}	V_{45}	Density
	km/sec	km/sec	km/sec	km/sec	km/sec	g/cc
	1.435	2.686	1.108	1.697	1.164	3.164
# of Reps	3	6	8	3	4	
% Std Dev	0.362	1.047	0.570	0.458	1.585	

ELASTIC CONSTANTS

C_{11}	C_{22}	C_{33}	C_{44}	C_{55}	C_{66}	C_{12}	C_{13}
GPa	GPa	GPa	GPa	GPa	GPa	GPa	GPa
22.82	22.82	6.52	3.89	3.89	9.11	4.59	1.86

ENGINEERING CONSTANTS

	GPa	Ksi		
Ex=Ey	21.55	3124.3	Young' Modulus(in-Plane)	
$\nu_{12}=\nu_{21}$	0.18	0.18	Poisson's Ratio	
$E_z=E_3$	6.26	907.9	Young's Modulus (out-of Plane)	
$\nu_{31}=\nu_{32}$	0.07	0.07	Poisson's Ratio	
$\nu_{23}=\nu_{13}$	0.23	0.23	Poisson's Ratio	
$G_{13}=G_{23}$	3.89	563.5	Shear Modulus(in-Plane)	
G_{12}	9.11	1321.7	Shear Modulus (out-of-Plane)	

11. TEST PROCEDURE (LOAD-DEPENDENCE AND VARIABILITY)

The velocity for the ultrasonic modes propagating parallel to the “3” axis, (V_{33}, V_{32}, V_{31}), exhibit significant variation with load. It is of value to quantify the load-dependence of these modes. The measurement procedure is similar to that described in section 9.

11.1 Attach the pair of longitudinal transducers in the load fixture as indicated in Figure 6.

11.2 Couple ultrasound into the sample and load to maximum coupling force (typically) 1000 Newtons.

11.3 Identify measurement point and monitor transit time changes as load is reduced to 100 Newtons.

11.4 Calculate the velocity V_{33} and coupling pressure.

11.5 Repeat measurements for 3 rectangular pieces.

11.6 Attach the pair of shear transducers in the load fixture as indicated in Figure 6.

11.7 Repeat steps 11.2 to 11.5 for the V_{32} mode.

12. RECOMMENDED INFORMATION TO INCLUDE IN LOAD DEPENDENCE REPORT

12.1 Same background information as indicated in 10.1 and 10.2.

12.2 Raw Data

TABLE 3 - LOAD DEPENDENCE OF VELOCITY AND ELASTIC CONSTANTS

Sample ID: AAA			Sample Thickness: 8.636 mm				
Load	T_{33}	T_{32}	Pressure	V_{33}	V_{32}	%Mod C_{33}	%Mod C_{44}
(N)	(µsec)	(µsec)	(MPa)	(km/sec)	(km/sec)		
1000	5.20	9.76	5.83	1.822	0.934	0.0	0.0
900	5.20	9.76	5.25	1.822	0.934	0.0	0.0
700	5.26	9.88	4.08	1.799	0.922	-2.5	-2.5
500	5.40	10.06	2.91	1.748	0.904	-7.9	-6.2
300	5.62	10.32	1.75	1.674	0.880	-15.6	-11.1
200	5.80	10.56	1.17	1.617	0.859	-21.2	-15.3
100	6.04	10.80	0.58	1.548	0.839	-27.8	-19.2

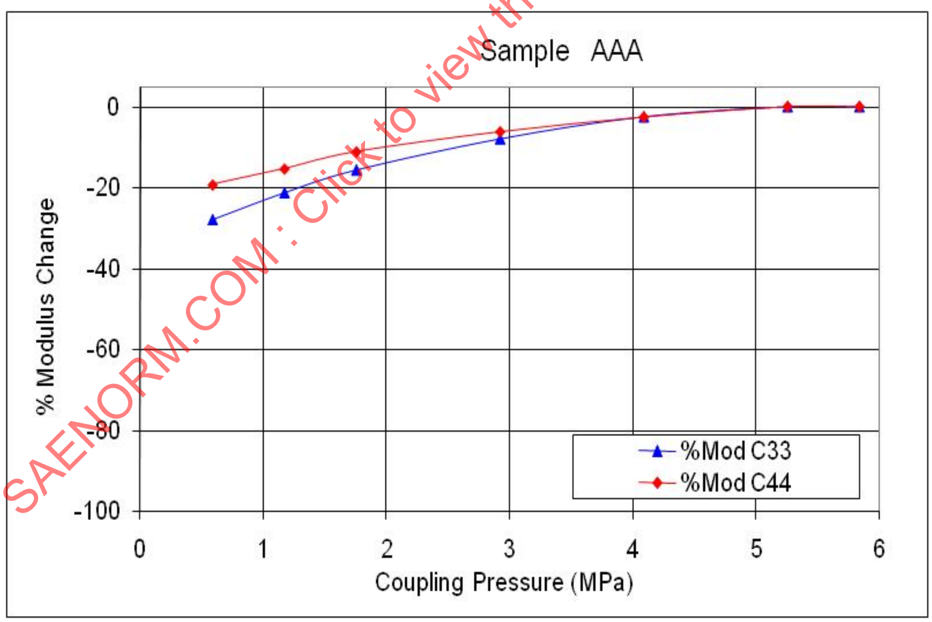

FIGURE 8 - LOAD DEPENDENCE OF ELASTIC CONSTANTS

Above table and graphics show the results obtained on a single sample. We recommend that a minimum of 3 samples should be measured.

13. NOTES

13.1 Marginal Indicia

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APPENDIX A - THEORY: DETERMINATION OF ELASTIC PROPERTIES FROM ULTRASONIC MEASUREMENTS

A.1 INTRODUCTION

Ultrasonic methods have been used for the past forty years to determine the elastic constants in anisotropic materials. There exists a fundamental relationship among sound speed in specific propagation directions and the elastic constants. For the most part this work has been carried out in physical acoustic laboratories on single crystals where the symmetry of the crystal is known. The extraction of elastic constant and subsequently the engineering constants is somewhat complex and depends on the symmetry of the sample being tested.

The purpose of this note is to describe the methods, which are used to extract elastic constants from materials, which possess transverse isotropy. We will first establish nomenclature for variables, constants and the coordinate system, then derive the governing relations between velocities of elastic waves propagation, physical elastic constants and engineering elastic constants.

A.2 NOMENCLATURE, COORDINATE SYSTEM, AND GOVERNING RELATIONS

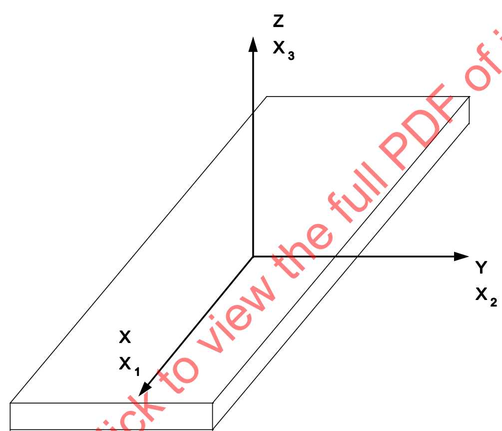

FIGURE A1 - COORDINATE DEFINITION

The coordinate system is shown in Figure A1. The axis coordinates (xyz) and ($x_1x_2x_3$) are used interchangeably. For transversely isotropic material the unique axis is $z \equiv x_3$.

It is known from the elementary theory of elasticity, that for small deformations the elongation components ε_x, ε_y, ε_z, and the shear angle components γ_{xy}, γ_{xz}, and γ_{yz} are given by

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \qquad \varepsilon_y = \frac{\partial u_y}{\partial y} \qquad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

(Eq. A1)

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \qquad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

where

u_x, u_y, u_z are displacements in directions x, y, z, respectively.

The strain tensor, i.e., the Green-St Venant tensor is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad \text{(Eq. A2)}$$

Substituting Equations (A1) in (A2) we can write the Green-St Venant tensor in its matrix representation as

$$[\varepsilon_{xyz}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} \qquad \text{(Eq. A3)}$$

Tensor (A3) is symmetric, which is obvious when we consider that reversing indices in (A2) does not change the equation. Because of this symmetry, there are only 6 independent components of tensor (A3). Notation is greatly simplified if we define the following six strain components:

$$\begin{aligned} \varepsilon_1 &= \varepsilon_x \\ \varepsilon_2 &= \varepsilon_y \\ \varepsilon_3 &= \varepsilon_z \\ \varepsilon_4 &= \gamma_{xz} \\ \varepsilon_5 &= \gamma_{yz} \\ \varepsilon_6 &= \gamma_{xy} \end{aligned} \qquad \text{(Eq. A4)}$$

The simplified stress components can be defined in an analogous way:

$$\begin{aligned} \sigma_1 &= \sigma_x \\ \sigma_2 &= \sigma_y \\ \sigma_3 &= \sigma_z \\ \sigma_4 &= \tau_{xz} \\ \sigma_5 &= \tau_{yz} \\ \sigma_6 &= \tau_{xy} \end{aligned} \qquad \text{(Eq. A5)}$$

The constitutive relations for an anisotropic material can be written using (A4) and (A5) in a simple matrix form.

It can be shown that the constitutive relations for an anisotropic material can be written using a simple matrix notation, instead of tensor notation:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \quad \text{(Eq. A6)}$$

Matrix [c] is symmetric therefore 21 constants are needed to fully describe elastic properties of the material.

A transversely isotropic material has the following properties:

– reversing direction of any of the axis of the coordinate system does not change matrix [c].
– if *x*-*y* plane is isotropic, then replacing *x* and *y*, does not change matrix [c] in (A1).

We need to remember, that matrix Equation (A6) is valid only in the coordinate system shown in Figure 1. Using these symmetry arguments it can be shown that

$$C_{14} = 0 \quad C_{24} = 0 \quad C_{34} = 0 \quad C_{45} = 0$$

$$C_{16} = 0 \quad C_{26} = 0 \quad C_{36} = 0 \quad C_{56} = 0 \quad \text{(Eq. A7)}$$

and

$$c_{11} = c_{22}, \qquad c_{13} = c_{23} \qquad c_{44} = c_{55} \quad \text{(Eq. A8)}$$

Since the *x*-*y* plane is isotropic, we can write the planar version of Hooke's law for this plane as:

$$c_{66} = \frac{E}{2(1+\nu)} = \frac{E(1-\nu)}{2(1+\nu)(1-\nu)} = \frac{E - E\nu}{2(1-\nu^2)} = \quad \text{(Eq. A9)}$$

$$= \frac{1}{2}\left(\frac{E}{1-\nu^2} - \frac{E\nu}{1-\nu^2}\right) = \frac{1}{2}(c_{11} - c_{12})$$

We can show that the matrix formulation of constitutive relations for the transversely isotropic material with unique *z* axis becomes:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11} - c_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xz} \\ \gamma_{yz} \\ \gamma_{xy} \end{bmatrix} \quad \text{(Eq. A10)}$$

Similarly, the compliance relationship can be obtained as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xz} \\ \gamma_{yz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11} - s_{12}) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{xy} \end{bmatrix} \quad \text{(Eq. A11)}$$

where

[c][s] = [I].